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On the theory of discrete structural model analysis and design

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Abstract

Exclusive theory for analysis of Structural Models (comprising of springs, masses, dash pots, etc.) is presented by adapting the electrical network theory. It commences a brief statement of a new Principle of Quasi Work (PQW), relevant to this theory. Derivations presented here include theorems addressing maximum displacements, relative flexibilities, sensitivity analysis of global flexibilities, inverse problem of load prediction and interpolation of stiffnesses and flexibilities of the Structural Models. Finally a 'Design Equation' capable of providing a starting point which more or less satisfies all the displacement constraints for iterative design employing a pair of estimated starting points for design iterations (within or outside feasible region) is evolved. Simple substantive illustrations are included to demonstrate the potential of these theoretical developments.

1. Introduction

Recognizing the potentials of the development of simplified structural models both for analysis and design (Harris and Crede, 1988; Prasad and Subba Rao, 1989; Prasad, 1992) provided the motivation for the research (Panditta, 1996) on new simplified analytical tools for these models.

In the field of electrical networks, topology is considered as the basis for defining the skeletal form of networks and Tellegen's theorem (Paul et al., 1970) is utilized for analyzing Topologically Similar Systems (TSS). As topology of structural models is similar to that of electrical networks; concept of TSS and Tellegen's theorem are adapted from the field of electrical network into the realm of structural mechanics resulting in a new energy principle termed as the Principle of Quasi Work (PQW). The derivation of the principle, its wide applicability and potentials relative to the existing energy theorems are being concurrently published. Hence, the statement of the theorem (without proof) and the explanation of the basic concept of 'TSS' is included here briefly for the sake of completeness.

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If two or more structures (such as, for example a 10 storeyed building and a 100 storeyed building) can be modelled by an identical set of nodes and branches (even if the branches contain null elements), these can be termed as topologically similar systems. (In this example, the top non-existent 90 storeys in the first system contain elements with zero stiffness/mass etc.) Moreover, if the constraints are changed in any structural system (by restraining some of the degrees of freedom), the basic topology is not altered. Hence, even such variants can be termed as 'TSS'. In essence, all the sets of such group of structures should be governed by a common set of mathematical relations say $[K]\{\rho\} = \{P\}$, where $[K]$ is construed to have an identical size (of say 100×100) for all the variants. Thus in the example, cited above, of 10 storeyed and 100 storeyed buildings, stiffness matrix of the first TSS has only rows 1–10 and columns 1–10 as non-zero elements, yet the pair of systems are termed as topologically equivalent.

To facilitate addressing various aspects of simplified analysis of models, a number of these theorems (having their genesis in the network analysis) are derived utilizing stiffness/flexibility coefficients corresponding to a pair of generalized directions (each of which is pertinent to a global DOF). A generalized direction is termed as an active direction only if an external force is acting in that direction, otherwise, it is termed as a passive direction. Further, as structural systems are modelled by elements having multiple (say, N_D) DOFs/node, the 'N' DOF of the system can be subdivided into N_D groups with only one corresponding DOF from each of its nodes. To facilitate ease in illustration of these theorems pertinent to static analysis of structural model comprising of discrete structural element models (forming the lower end of the element spectrum of FEM); a typical set of structural models (with a common topological layout, see Fig. 1(A)) and having single DOF/node are chosen.

Thus, following sections present the theory for the simplified analysis of discrete structural models together with substantive illustrations and the derivation of a 'Design Equation' capable of providing a starting design point for problems with active displacement constraints.

2. The Principle of Quasi Work (PQW)

In a pair of TSS (*m and n*), *quasi work done by* (*self equilibrating system of*) *external forces of* any one of the systems *while going through the corresponding* (*compatible*) *displacements of the* other system, *is equal to quasi energy due to internal forces of former system while going through corresponding deformations of the latter system.*

In this context it may be relevant to state that the concept of branches is borrowed from electrical field. However, in structural systems one first identifies the lines of load flow. Each path of low flow is termed as branch. This branch can have series of simple as well as compound elements. Each element can have internal force and deformation (relative to its terminal nodes) which can be utilised to obtain the strain energy stored.

It may be relevant to note that Quasi Work and Quasi Energy expressions relate to a pair of TSS '*m*' and '*n*' (as per definitions given in Tables 1 and 2, respectively) and the application of PQW to these systems yields:

$$U_{mn} = W_{mn} \quad \text{or } \{F\}_m^T\{\delta\}_n = \{P\}_m^T\{d\}_n \tag{1}$$

where

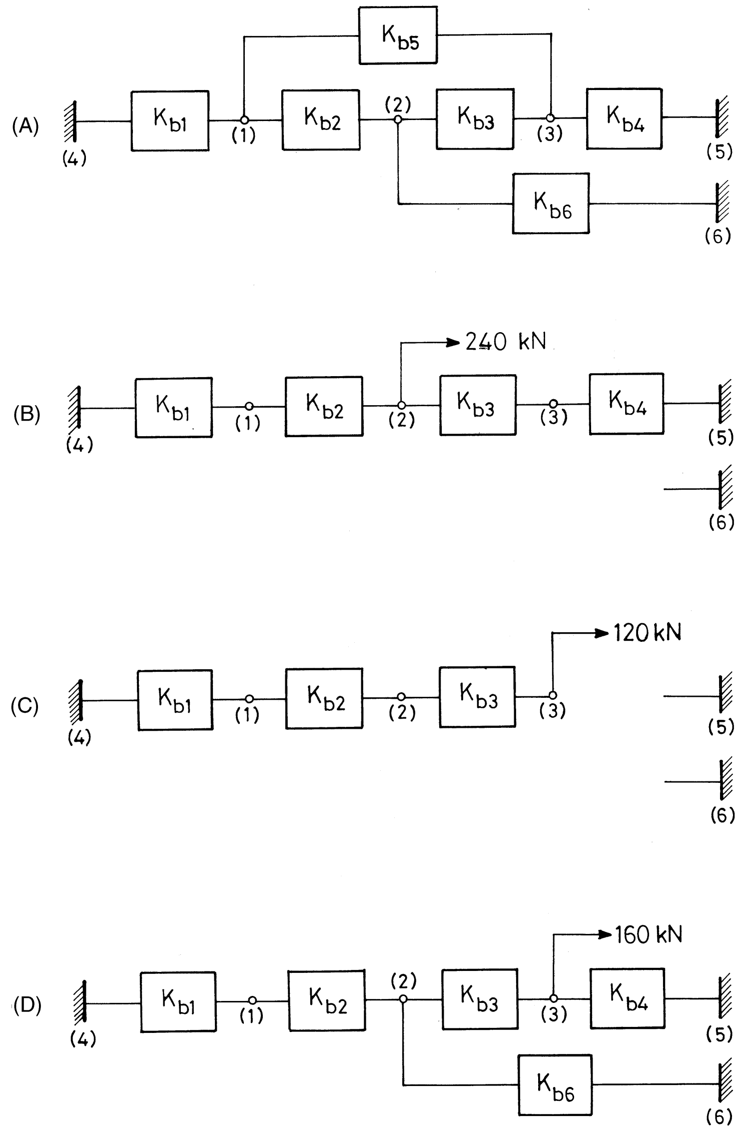

Fig. 1. A set of topologically similar systems. (A) Topological layout of a six element model. (B) (TSS_1). (C) (TSS_2). (D) (TSS_3).

$\{F\}_m^T$ is internal branch forces of TSS_m;
$\{d\}_n$ is displacement vector of TSS_n;
$\{P\}_m^T$ is external load applied on TSS_m;
$\{\delta\}_n$ is branch deformation vector of TSS_n;
$U_{mn}(=\{F\}_m^T\{\sigma\}_n)$ is quasi energy corresponding to the TSS pair, TSS_m and TSS_n, and
$W_{mn}(=\{P\}_m^T\{d\}_n)$ is quasi work corresponding to this TSS pair.

Table 1
Quasi work computations

Node number		1	2	3	4	5	6
$(TSS)_1$	$\{P\}_1^T$	0	240	0	-120	-120	0
	$\{d\}_1^T$	6	10	6	0	0	0
$(TSS)_2$	$\{P\}_2^T$	0	0	120	-120	0	0
	$\{d\}_2^T$	4	8	12	0	0	0
$(TSS)_3$	$\{P\}_3^T$	0	0	160	-40	-80	-40
	$\{d\}_3^T$	1	2	4	0	0	0
$W_{12} = \{P\}_1^T\{d\}_2 = 1920$		0	$+1920$	$+0$	$+0$	$+0$	$+0$
$W_{21} = \{P\}_2^T\{d\}_1 = 720$		0	$+0$	$+720$	$+0$	$+0$	$+0$
$W_{13} = \{P\}_1^T\{d\}_3 = 480$		0	$+480$	$+0$	$+0$	$+0$	$+0$
$W_{31} = \{P\}_3^T\{d\}_1 = 960$		0	$+0$	$+960$	$+0$	$+0$	$+0$
$W_{23} = \{P\}_2^T\{d\}_3 = 480$		0	$+0$	$+480$	$+0$	$+0$	$+0$
$W_{32} = \{P\}_3^T\{d\}_2 = 1920$		0	$+0$	$+1920$	$+0$	$+0$	$+0$

$(TSS)_1: K_{b1} = K_{b4} = 20;\quad K_{b2} = K_{b3} = 30;\quad K_{b5} = K_{b6} = 0$
$(TSS)_2: K_{b1} = K_{b2} = K_{b3} = 30;\quad K_{b4} = K_{b5} = 0;\quad K_{b6} = 0$
$(TSS)_3: k_{b1} = K_{b2} = K_{b3} = 40;\quad K_{b4} = K_{b6} = 20;\quad k_{b5} = 0$
All stiffness coefficients are in kN/mm

Interchanging the systems in eqn (1) and subtracting the resulting equation from eqn (1), we obtain:

$$\{F\}_m^T\{\delta\}_n - \{F\}_n^T\{\delta\}_m = \{P\}_m^T\{d\}_n - \{P\}_n^T\{d\}_m \tag{2}$$

Substantive Illustration. Figure 1(A) represents the common topology for a set of structural model systems $(TSS)_1$, $(TSS)_2$ and $(TSS)_3$ (see Figs 1(B)–(D). Computations of Quasi Work and Quasi Energy are detailed in Tables 1 and 2, respectively, and illustrate the PQW.

3. Maximum displacement directions

This is a theorem which can help in identifying global directions where maximum displacement can occur, which in turn aids a designer to do away with irrelevant displacement constraints.

Statement: Maximum displacement in any passive direction (belonging to a group of generalized

Table 2
Quasi energy computations

Branch number		1	2	3	4	5	6
$(\mathrm{TSS})_1$	$\{F\}_1^T$	120	120	-120	-120	0	0
	$\{\delta\}_1^T$	6	4	-4	-6	0	-10
$(\mathrm{TSS})_2$	$\{F\}_2^T$	120	120	120	0	0	0
	$\{\delta\}_2^T$	4	4	4	-12	8	-8
$(\mathrm{TSS})_3$	$\{F\}_3^T$	40	40	80	-80	0	-40
	$\{\delta\}_3^T$	1	1	2	-4	3	-2
$U_{12} = \{F\}_1^T\{\delta\}_2 = 1920$		480	$+480$	-480	$+1440$	$+0$	$+0$
$U_{21} = \{F\}_2^T\{\delta\}_1 = 720$		720	$+480$	-480	$+0$	$+0$	$+0$
$U_{13} = \{F\}_1^T\{\delta\}_3 = 480$		120	$+120$	-240	$+480$	$+0$	$+0$
$U_{31} = \{F\}_3^T\{\delta\}_1 = 960$		240	$+160$	-320	$+480$	$+0$	$+400$
$U_{23} = \{F\}_2^T\{\delta\}_3 = 480$		120	$+120$	$+240$	$+0$	$+0$	$+0$
$U_{32} = \{F\}_3^T\{\delta\}_2 = 1920$		160	$+160$	$+320$	$+960$	$+0$	$+320$

From Tables 1 and 2: $U_{mn} = W_{mn}$ [for $(\mathrm{TSS})_m$ and $(\mathrm{TSS})_n$]

directions) *cannot exceed the maximum displacement among active directions* (*belonging to the same group*) *provided there are no transformable linkages.*

The term '*transformable linkage*' needs some explanation. In structural/mechanical systems consisting of levers, gears, etc. one comes across displacement and force transformations arising out of the linkages. In such systems, displacements/forces can be transformed in order that the existence of the link can be totally removed during analysis. Such linkages are termed as transformable linkages. For example, in electrical systems, the circuit preceeding the transformer and succeeding the transformer are referred to as primary and secondary sides of the transformer, respectively. Whereas, the circuit can be replaced by suitable arrangement of the elements to one of the sides only (on the basis energy equivalence). Similar situation arises even in structures/machines with levers/gears. This theorem excludes such scaled systems in the original form from its preview of finding out the maximum displacement, etc. However, in the case of such systems, this theorem is definitely applicable to any of its equivalent systems derived (based on energy equivalence) after removing all such linkages through appropriate transformations of all relevant elements a priori.

Proof: (*Reduxo Absurdum*). Consider a model, $(\mathrm{TSS})_m$, having a single group of DOF obtained by suitably restricting DOF of each node to one (say for example, axial direction) and a given set of applied loads acting in some of these directions. Due to this simplification, active directions

Table 3
Correlations between active and maximum displacement directions

Loads (kN)			Displacements (mm)		
P_1	P_2	P_3	d_1	d_2	d_3
<u>30</u>	0	0	<u>0.421</u>	0.142	0.203
0	<u>40</u>	0	0.189	<u>0.518</u>	0.218
0	0	<u>50</u>	0.338	0.272	<u>0.763</u>
<u>60</u>	0	50	<u>1.179</u>	0.556	1.168

$K_{b1} = 2K_{b2} = 40$ kN/mm; $2K_{b3} = K_{b6} = 50$ kN/mm
$K_{b4} = 30$ kN/mm; $K_{b5} = 35$ kN/mm

become synonymous with active nodes. Assume for the sake of argument that a passive node 'i' ($P_{im} = 0$) has displacement more than the maximum displacement among the active nodes. Choose a $(\text{TSS})_n$ with all nodes except node 'i' as fixed (i.e. $\{d\}_n$ has all entries zero except the component d_{in} which is equal to unity). Hence, from eqn (1), we have:

$$\{F\}_m^T\{\delta\}_n = \{P\}_m\{d\}_n = 0 \tag{3}$$

Only branches connected to node 'i' will contribute to the left hand side of the above equation. As node 'i' has maximum displacement relative to all the adjacent nodes in both the TSS, the signs of corresponding elements of $\{\delta\}_n$ and $\{F\}_n$ are the same which also correspond to those of $(\text{TSS})_m$ for all the branches connected to this node. Hence, the expression

$$\{F\}_m^T\{\delta\}_n > 0 \tag{4}$$

This leads to a contradiction, falsifying our assumption. Thus, at a passive node maximum displacement cannot exceed the maximum value occurring at active nodes, provided left hand side of eqn (3) does not vanish (which forms the exceptional case stated in the theorem). Hence, the theorem is proved for systems with one DOF per node. However, if it has more than one DOF per node, the theorem can be proved for each group of DOF separately.

This theorem may be effectively utilized to evolve a conservative design by replacing the displacement constraints in all active directions belonging to a group by the least value among the displacement constraints of the corresponding group of generalized directions (since, constraints in all other non-active directions can never be violated and become irrelevant).

Illustration: The nodal displacements in the example problem together with the chosen set of branch stiffness parameters are given in Table 3 for four load cases. The bold underlined values illustrate the occurrence of maximum deflection in one of the active directions only.

4. Relative and conditional flexibilities

The following concept of relative flexibility is useful for evaluating relative strengths in global directions of a structural model and helps in identifying the most/least flexibility directions.

Relative flexibility, $\eta_{i/j}$, for a pair of directions 'i' and 'j', is defined as the ratio of their diagonal flexibility coefficients f_{ii} and f_{jj}.

This section is devoted to a theorem pertaining to the simplified calculation of relative flexibilities through evaluation of conditional flexibilities.

The term conditional flexibility, $f_{ii/j}$, refers to the flexibility f_{ii} obtained for a TSS created from a given model (of a structural system) by completely restraining the direction 'j'.

Statement of the theorem: The relative flexibility, $\eta_{i/j}$, of any pair of directions, 'i' and 'j', is equal to the ratio of their respective conditional flexibilities, '$f_{ii/j}$' and '$f_{jj/i}$', viz

$$\eta_{i/j} = \frac{f_{ii}}{f_{jj}} = \frac{f_{ii/j}}{f_{jj/i}} \tag{5}$$

Proof: In order to facilitate independent definitions of all the four flexibility coefficients in eqn (5), consider four different TSS ('p', 'q', 'r' and 's') with identical branch properties but having different load/fixity conditions in directions 'i' and 'j' as under:

I: $(\mathrm{TSS})_p$, in which P_{ip} is acting and $P_{jp} = 0$. Hence,

$$f_{ii} = d_{ip}/P_{ip} \tag{6}$$

II: $(\mathrm{TSS})_q$, in which P_{jq} is acting and $P_{iq} = 0$. Hence,

$$f_{jj} = d_{jq}/P_{jq} \tag{7}$$

III: $(\mathrm{TSS})_r$, in which P_{ir} is acting and $d_{jr} = 0$. Hence,

$$f_{ii/j} = d_{ir}/P_{ir} \tag{8}$$

IV: $(\mathrm{TSS})_s$, in which only P_{js} is acting and $d_{is} = 0$. Hence,

$$f_{jj/i} = d_{js}/P_{js} \tag{9}$$

Applying eqn (2) for the systems with active nodes, 'i' and 'j', we have:

$$P_{im}d_{in} - P_{in}d_{im} + P_{jm}d_{jn} - P_{jn}d_{jm} = \{F\}_m^T\{\delta\}_n - \{F\}_n^T\{\delta\}_m \tag{10}$$

The right hand side in this equation vanishes as $\{F\} = \{\delta\}^T[f_b]^{-1}$ and matrix $[f_b]$ is the same for all the four systems. Hence, eqn (10) reduces to:

$$P_{im}d_{in} - P_{in}d_{im} + P_{jm}d_{jn} - P_{jn}d_{jm} = 0 \tag{11}$$

Replacing 'm' and 'n' successively in the above equation for the following four pairs of TSS: by (a) 'p' and 'q', (b) 'r' and 's', (c) 'p' and 'r' and (d) 'q' and 's', respectively, and utilizing the conditions as well as eqns (6)–(10) stated in defining the systems p, q, r, s; result in the following four equations.

$$P_{ip}d_{iq} = P_{jq}d_{jp} \tag{12}$$

$$P_{jr}d_{js} = P_{is}d_{ir} \tag{13}$$

$$P_{ip}d_{ir} = P_{ir}d_{ip} + P_{jr}d_{jp} \tag{14}$$

$$P_{jq}d_{js} = P_{is}d_{iq} + P_{js}d_{jq} \tag{15}$$

Dividing eqn (13) by eqn (12) and rearranging the terms we get:

$$\frac{P_{jr}d_{jp}}{P_{ip}d_{ir}} = \frac{P_{is}d_{iq}}{P_{jq}d_{js}} \tag{16}$$

Substituting eqns (14), (15) in eqn (16) and utilizing the flexibility coefficients from eqns (6)–(9), we get:

$$\frac{f_{ii}}{f_{ii/j}} = \frac{f_{jj}}{f_{jj/i}} \tag{17}$$

Which leads to eqn (5), proving the theorem for relative flexibilities.

Illustration: In view of the importance of this theorem, we shall consider the most general case of the example problem and provide the illustration algebraically to lend credibility to its power and use.

The diagonal, conditional and relative flexibilities for the model (Fig. 1(A)) are presented in Table 4. The entries in rows 5–7 illustrate the theorem on relative flexibilities (except where conditional flexibilities are zero, in which case the result is obvious).

5. Inverse problem of load prediction

Here, a computationally efficient theorem for the inverse problem of determination of loads in all directions which can result in a given distribution of element forces, is stated and proved. This is useful for comparative evaluation of models for their load carrying capabilities and for computing safe limit for external loading for a model with prescribed or computed (from material and geometry) set of maximum permissible branch forces.

Statement of the theorem: Applied force in any active direction 'j' of a model, (TSS), *which can result in a given distribution of branch forces, F, is given by:*

$$P_{jm} = \{F\}_m\{\bar{\delta}\}_n \tag{18}$$

where, δ is the non-dimensional deformation vector corresponding to the unit displacement in direction 'j' in (TSS) *which is derived from the model by fixing it in all other directions.*

Proof: As all $\{d\}_n$ are zero concept d_{jn} which is equal to unity, the right hand side of eqn (1) reduces to P_{jm}. Left hand side of the equation remains the same except that the branch deformations correspond to unit displacement i.e. branch deformation vector $\{\delta\}_n$ takes its non-dimensional form $\{\bar{\delta}\}_n$. Hence, eqn (1) takes the form of eqn (18) on rearrangement.

Illustration: For demonstrating the estimation of the loads through inverse process, first of all the direct problem corresponding to arbitrary load vector, $\{P_m\}$, is solved for the example problem resulting in displacement vector, $\{d\}_m$, and branch force vector $\{F\}_m$ as given in part I of Table 5. In fact, if the branch force vector corresponds to proof loads (determined for individual

Table 4
Computations of relative flexibilities

Sr. no.	Flexibility coefficients	Nodes (i) 1	2	3
1	f_{ii}	$\dfrac{BC-K_{b3}^2}{D}$	$\dfrac{AC-K_{b5}^2}{D}$	$\dfrac{AB-K_{b2}^2}{D}$
2	$f_{ii/1}$ Node 1 is fixed	0	$\dfrac{C}{BC-K_{b3}^2}$	$\dfrac{B}{BC-K_{b3}^2}$
3	$f_{ii/2}$ Node 2 is fixed	$\dfrac{C}{AC-K_{b5}^2}$	0	$\dfrac{C}{AC-K_{b5}^2}$
4	$f_{ii/3}$ Node 3 is fixed	$\dfrac{B}{AB-K_{b2}^2}$	$\dfrac{A}{AB-K_{b2}^2}$	0
5	$\eta_{i/1}=\dfrac{f_{ii}}{f_{11}}=\dfrac{f_{ii/1}}{f_{11/i}}$	1	$\dfrac{AC-K_{b5}^2}{BC-K_{b3}^2}$	$\dfrac{AB-K_{b2}^2}{BC-K_{b3}^2}$
6	$\eta_{i/2}=\dfrac{f_{ii}}{f_{22}}=\dfrac{f_{ii/2}}{f_{22/i}}$	$\dfrac{BC-K_{b3}^2}{AC-K_{b5}^2}$	1	$\dfrac{AB-K_{b2}^2}{AC-K_{b5}^2}$
7	$\eta_{i/3}=\dfrac{f_{ii}}{f_{33}}=\dfrac{f_{ii/3}}{f_{33/i}}$	$\dfrac{BC-K_{b3}^2}{AB-K_{b2}^2}$	$\dfrac{AC-K_{b5}^2}{AB-K_{b2}^2}$	1

where
$A = K_{b1}+K_{b2}+K_{b5}$; $B = K_{b2}+K_{b3}+K_{b6}$; $C = K_{b3}+K_{b4}+K_{b5}$
$D = ABC-AK_{b3}^2-BK_{b5}^2-CK_{b2}^2-2K_{b2}^2K_{b3}^2K_{b5}^2$

branch/elements based on their material characteristics) then this theorem will yield safe external loading for the model.

Part II of the table illustrates the above theorem for computing nodal load, P_{jm}, corresponding to the branch force vector $\{F\}_m$. Branch deformations $\{\bar{\delta}\}_n$ correspond to unit displacement mode for node 'j' of $(\text{TSS})_n$, reckoned positive if tensile. This constitutes the inverse solution of load estimation which when compared with the load vector given in Part I completes the illustration.

6. Interpolation of linear systems

In this section, the problem of interpolation of two linear systems having only stiffness elements is considered in order to facilitate creation of better estimated pair of starting points for design. The properties of global stiffness and diagonal flexibility coefficients are investigated.

For this purpose, let a discrete structure $(\text{TSS})_m$ be created through generation of its branch

Table 5
Illustration of load estimation

Part I: Direct solution

Assumed applied load $\{P\}_m^T = \{-50 \quad 95 \quad 185\}$ kN
Assumed system parameters:
$K_{b1} = 2K_{b2} = 40$ kN/mm; $K_{b4} = 30$ kN/mm
$K_{b6} = 2K_{b3} = 50$ kN/mm; $K_{b5} = 35$ kN/mm
Corresponding displacement vector: $\{d\}_m^T = \{1 \quad 2 \quad 3\}$ mm
and branch forces $\{F\}_m^T = \{40 \quad 20 \quad 25 \quad -90 \quad 70 \quad -100\}$ kN

Part II: Inverse process of load estimation of $(\mathrm{TSS})_m$

Branch data			$\{\bar{\delta}\}_n$ for a unit displacement at node 'j' with other nodes fixed		
No. k	Nodes Nos	Forces F_k (kN)	$\bar{\delta}_k$ ($j = 1$)	$\bar{\delta}_k$ ($j = 2$)	$\bar{\delta}_k$ ($j = 3$)
1	4–1	40	1	0	0
2	1–2	20	−1	1	0
3	2–3	25	0	−1	1
4	3–5	−90	0	0	−1
5	1–3	70	−1	0	1
6	2–6	−100	0	−1	0
$P_{jm} = \{F\}_m^T \{\bar{\delta}\}_n \Rightarrow$			−50 kN	95 kN	185 kN

stiffnesses by a linear interpolation of the corresponding branch stiffness of a pair of topologically similar discrete structures, $(\mathrm{TSS})_p$ and $(\mathrm{TSS})_q$ as under:

$$(K_{bk})_m = [\alpha(K_{bk})_p + \beta(K_{bk})_q]/(\alpha+\beta) \tag{19}$$

where, α and β are arbitrary positive constants.

For such an interpolation, the relationships which govern the respective stiffness and diagonal nodal flexibility coefficients are given by:

Statement I (*for stiffness coefficients*)*: The global stiffness parameters for any* TSS *generated through linear interpolation of a given pair of* TSS, *are equal to the corresponding interpolated values of their global stiffness parameters.*

$$(K_{ij})_m = [\alpha(K_{ij})_p + \beta(K_{ij})_q]/(\alpha+\beta) \tag{20}$$

This is a linear interpolation achieved through 'assembly' and is too obvious to be proved.

Statement II (*for diagonal flexibility coefficients*)*: If a* TSS *is generated through a linear interpolation of a given pair of* TSS*; the lower bound for its reciprocal diagonal flexibility coefficient in any of the directions is given by the interpolated value of corresponding reciprocal coefficients of the pair*, *i.e.*

$$(1/f_{jj})_m \geqslant [\alpha(1/f_{jj})_p + \beta(1/f_{jj})_q]/(\alpha+\beta) \tag{21}$$

Proof: Considering the action of external force in a direction '*j*', for $(\mathrm{TSS})_m$, equation relating strain energy and external work done is given by:

$$\{P\}_m^T\{d\}_m = \{F\}_m^T\{\delta\}_m \tag{22}$$

which on reduction yields (as $P_j = d_j/f_{jj}$ and $F_k = K_{bk}\delta_k$):

$$(1/f_{jj})_m = \sum_{k=1}^{B}\left(\frac{\delta_{km}}{d_{jm}}\right)^2 (K_{bk})_m \tag{23}$$

where, B is the total number of branches.

Similarly, for $(\mathrm{TSS})_p$, we obtain

$$(1/f_{jj})_p = \sum_{k=1}^{B}\left(\frac{\delta_{kp}}{d_{jp}}\right)^2 (K_{bk})_p \tag{24}$$

Application of PQW for $(\mathrm{TSS})_p$ and $(\mathrm{TSS})_m$ [i.e. eqn (1)] on reduction yields

$$(1/f_{jj})_p = \sum_{k=1}^{B}\frac{\delta_{kp}}{d_{jp}}\frac{\delta_{km}}{d_{jm}}(K_{bk})_p \tag{25}$$

Consider the following inequality arising from the summation of positive definite terms:

$$\sum_{k=1}^{B}\left[\frac{\delta_{km}}{d_{jm}} - \frac{\delta_{kp}}{d_{jp}}\right]^2 (K_{bk})_p \geqslant 0 \tag{26}$$

which on substituting from eqns (24), (25) and after scaling with a factor 'α', yields:

$$\sum_{k=1}^{B}\left(\frac{\delta_{km}}{d_{jm}}\right)^2 \alpha(K_{bk})_p \geqslant \alpha(1/f_{jj})_p \tag{27}$$

Similarly, considering the pair $(\mathrm{TSS})_q$ and $(\mathrm{TSS})_m$, we have,

$$\sum_{k=1}^{B}\left(\frac{\delta_{km}}{d_{jm}}\right)^2 \beta(K_{bk})_q \geqslant \beta(1/f_{jj})_q \tag{28}$$

Adding eqns (27), (28) and utilizing eqns (19)–(23) yield the required result.

Statement II: If a TSS *is generated through a linear interpolation of a given pair of* TSS*; the upper bound for its diagonal flexibility coefficient in any of the directions is given by the reciprocal of the interpolated value of reciprocals of corresponding coefficients of the pair.*

$$(f_{jj})_m \leqslant (\alpha+\beta)[(f_{jj})_p(f_{jj})_q]/[\alpha(f_{jj})_q+\beta(f_{jj})_p] \tag{29}$$

Proof: This is a direct consequence of rearranging the reciprocal of eqn (21).

Illustration: For the common topological layout given in Fig. 1(A); $(\mathrm{TSS})_p$, $(\mathrm{TSS})_q$ and proportionality constants α and β associated with these two systems are first defined then $(\mathrm{TSS})_m$ is created by obtaining its branch elements through linear interpolation of corresponding branch elements of $(\mathrm{TSS})_p$ and $(\mathrm{TSS})_q$. Stiffness matrix coefficients for systems '*p*', '*q*', '*m*' and the

Table 6
Import of linear system interpolations

(a) Assembled stiffness coefficients and their interpolated values

System	K_{11}	K_{12}	K_{13}	K_{22}	K_{23}	K_{33}
$(\text{TSS})_p$	80	-20	-50	110	-30	120
$(\text{TSS})_q$	130	-50	-20	100	-40	90
$(\text{TSS})_m$	110	-38	-32	104	-36	102
Interpolated values with $\alpha = 4$ and $\beta = 6$						
$(\text{TSS})_m$	110	-38	-32	104	-36	102

(b) Parameters and reciprocal flexibility coefficients

System	K_{b1}	K_{b2}	K_{b3}	K_{b4}	K_{b5}	K_{b6}	$1/f_{11}$	$1/f_{22}$	$1/f_{33}$
$(\text{TSS})_p$	10	20	30	40	50	60	48.862	84.648	71.548
$(\text{TSS})_q$	60	50	40	30	20	10	83.378	54.602	58.762
$(\text{TSS})_r$	40	38	36	34	32	30	73.345	66.985	68.326
Interpolated values with $\alpha = 4$ and $\beta = 6$							69.572	66.631	63.874

interpolated values as per eqn (20) are given in Table 6(a). As can be seen from this table, the interpolated stiffness coefficients are identical with the corresponding assembled values.

Inverse diagonal flexibility coefficients for the three systems together with the corresponding interpolated values from systems 'p' and 'q' are given in Table 6(b). The values of the reciprocals of diagonal flexibility coefficients are less than the corresponding actual values for the system 'm', thereby, demonstrating the applicability of eqn (21). Further, it requires no further evidence, that the inequality change in respect of the comparison of flexibility coefficients of system 'm' and the interpolation values obtained from systems 'p' and 'q'. Hence, all the statements concerning interpolations stand substantiated.

7. Sensitivity evaluation

Sensitivity analysis is an important aspect of optimal design problems. Usually, these are viewed as either shape optimization (Kikuchi and Horimatsu, 1994; Haftka and Grandhi, 1986; Choi and Haug, 1983) for minimum weight or optimization for fully stressed design (Kikuchi and Horimatsu, 1994; Donald, 1994). In the documented methods on sensitivity, the gradients for displacements/stresses for a particular set of loading conditions are obtained by either finite difference (Khot, 1994; Haug et al. 1985; Brayton and Spence, 1985) or from principle of virtual forces (Akin, 1994; Donald, 1994). These coefficients do not truly represent the sensitivity of flexibility in a global direction, which should be independent of external loading.

In this section, a theorem which relates changes in the diagonal reciprocal flexibility coefficient in any particular direction due to changes in stiffness of a specific branch will be derived for two distinct cases. First case is pertinent to infinitesimal changes in branch stiffnesses and the second is pertinent to finite changes in branch stiffnesses.

Case 1 : *Infinitesimal changes in branch parameter*
A significant advantage of this part of the theorem lies in determination of sensitivity of flexibility coefficients in global directions corresponding to variations in branch stiffnesses, through a single solution to the equations governing the model.

Statement of the theorem: Partial derivative of the reciprocal of a generalized diagonal flexibility coefficient, 'f_{jj}', in any direction, 'j', with respect to the reciprocal of flexibility, 'f_{bk}', of any branch, 'k', is equal to the square of the ratio of the corresponding branch deformation, 'δ_k', and displacement, 'd_j', in the direction 'j'.
i.e.

$$\frac{\partial(1/f_{jj})}{\partial(1/f_{bk})} = \frac{\delta_k^2}{d_j^2} \tag{30}$$

which can also be written as:

$$\frac{\partial f_{jj}}{\partial f_{bk}} = \frac{f_{jj}^2 \delta_k^2}{f_{bk}^2 d_j^2} \tag{31}$$

Proof: Either from PQW [i.e. from eqn (1)] or independently from the equation relating the strain energy and the work done by external loads, we get

$$\{F\}^T\{\delta\} = \{P\}^T\{d\} \tag{32}$$

Replacing the external forces in terms of corresponding displacements and flexibilities and further restricting the loading to one direction ('j') only, this equation reduces to:

$$\{\delta\}^T[f_b]^{-1}\{\delta\} = d_j^2/f_{jj} \tag{33}$$

Recognizing the diagonal nature of $[f_b]$ and taking partial derivative of eqn (33) with respect to any branch flexibility f_{bk}, yields:

$$\frac{\partial}{\partial(1/f_{bk})}\left[\sum_{k=1}^{B}(\delta_k^2/f_{bk})\right] = \frac{\partial}{\partial(1/f_{bk})}[d_j^2/f_{jj}] \tag{34}$$

or

$$\delta_k^2 = d_j^2 \frac{\partial(1/f_{jj})}{\partial(1/f_{bk})} \tag{35}$$

since, d_j can be safely assumed not to vary due to infinitesimal changes in f_{bk}. This results in eqn (30) and, hence, eqn (31).

Table 7
Sensitivity of diagonal flexibility coefficients

(a) Load displacement data for the three load cases of basic system

Sr.	P_1	P_2	P_3	d_1	d_2	d_3	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6
1	100	0	0	2.05	0.65	1.01	2.05	−1.40	0.37	−1.01	−1.03	−0.65
2	0	200	0	1.30	2.36	1.13	1.30	1.06	−1.23	−1.31	−0.17	−2.36
3	0	0	150	1.52	0.85	2.10	1.52	−0.67	1.24	−2.10	0.57	−0.85

$K_{b1} = 10$; $K_{b2} = 20$; $K_{b3} = 30$: in kN/mm
$K_{b4} = 40$; $K_{b5} = 50$; $K_{b6} = 60$: in kN/mm

(b) Parameters of the basic and the perturbed systems

System	K_{b1}	K_{b2}	K_{b3}	K_{b4}	K_{b5}	K_{b6}	$1/f_{11}$	$1/f_{22}$	$1/f_{33}$
Basic	10.0	20.0	30.0	40.0	50.0	60.0	48.862	84.648	71.548
1	10.3	20.0	30.0	40.0	50.0	60.0	49.162	84.738	71.705
2	10.0	20.6	30.0	40.0	50.0	60.0	49.140	84.769	71.609
3	10.0	20.0	30.9	40.0	50.0	60.0	48.890	84.890	71.864
4	10.0	20.0	30.0	41.2	50.0	60.0	49.154	84.919	71.747
5	10.0	20.0	30.0	40.0	51.5	60.0	49.238	84.655	71.658
6	10.0	20.0	30.0	40.0	50.0	61.8	49.040	86.448	71.838

(c) Validation of flexibility gradient formula for infinitesimal changes

Direction (j)		Branch numbers: K 1	2	3	4	5	6
1	I	1.000	0.466	0.032	0.246	0.254	0.101
	II	1.000	0.463	0.031	0.243	0.251	0.099
2	I	0.301	0.203	0.272	0.229	0.005	1.000
	II	0.300	0.202	0.268	0.226	0.005	1.000
3	I	0.527	0.103	0.354	1.000	0.075	0.164
	II	0.523	0.102	0.351	0.999	0.074	0.161

$\mathrm{I} \rightarrow \delta_k^2/d_j^2$; $\mathrm{II} \rightarrow \partial(1/f_{jj})/\partial(1/f_{bk})$

Illustration: The process of illustrating sensitivity theorem for infinitesimal changes [i.e. eqn (30)] in reciprocal diagonal flexibility coefficients is presented for the example problem in Table 7. Table 7(a) presents the global displacement and branch deformation data of the basic system pertinent to three load cases of single load applications. The flexibility data needed for illustrating changes in each of the six branches are obtained by inverting the six stiffness matrices (generated by effecting 3% perturbation of the basic stiffness coefficients in only one of the branches at a time) are given in Table 7(b). While Table 7(c) provides comparison of flexibility gradients obtained from displacement data from Table 7(a) [i.e. case I basing on the right hand side expression of eqn (30)] and the flexibility data from Table 7(b) [i.e. case II, basing on left hand side expression of eqn (30)]. This establishes a sensitivity theorem, as the difference in these expressions is small.

Case 2: Finite changes in branch parameter

This part of the theorem will be useful in deriving a new Design Equation for problems involving displacement constraints. This equation is unique in the sense that with the use of this equation one can determine a design which closely (in some cases exactly!) satisfies the displacement constraints.

Statement of the theorem: The ratio of difference in the reciprocal of flexibility in any specific generalized direction between a pair of $(\mathrm{TSS})_m$ *and* $(\mathrm{TSS})_n$, *where* $(\mathrm{TSS})_n$ *is derived from* $(\mathrm{TSS})_m$ *by altering the flexibility of any particular branch, and the difference in the reciprocal of flexibility of the altered branch is equal to the ratio of the product of deformations in the chosen branch of the two* TSS *and the product of their generalized displacements in the chosen direction.*

$$\frac{1/(f_{jj})_m - 1/(f_{jj})_n}{1/(f_{bk})_m - 1/(f_{bk})_n} = \frac{\delta_{km}\delta_{kn}}{d_{jm}d_{jn}} \tag{36}$$

Proof: Assuming that the systems comprise of stiffness elements only and replacing forces by corresponding displacements and flexibilities when only direction 'j' is active, eqn (2) reduces to:

$$\{\delta\}_m^T[f_b]_m^{-1}\{\delta\}_n - \{\delta\}_n^T[f_b]_n^{-1}\{\delta\}_m = d_{jm}d_{jn}[1/(f_{jj})_m - 1/(f_{jj})_n] \tag{37}$$

If flexibility of only branch, 'k', is altered keeping the rest of branch flexibilities the same in both systems 'm' and 'n', then this equation reduces to:

$$(\delta_k)_m(\delta_k)_n[1/(f_{bk})_m - 1/(f_{bk})_n] = d_{jm}d_{jn}[1/(f_{jj})_m - 1/(f_{jj})_n] \tag{38}$$

which after rearranging the terms, takes the form of eqn (36), proving the statement. If the systems are obtained by minor perturbations, eqn (36) reduces to eqn (30) in the limit.

It may be relevant to note that the right hand side of eqn (38) represents differential Quasi Work expression:

$$(\Delta W)_{mn} = W_{mn} - W_{nm} \tag{39}$$

Illustration: The theorem is illustrated for the example problem by doubling the stiffness of its third branch. Results are presented in Table 8 in which entries in the last two columns are equal, thus completing the illustration.

Table 8
Flexibility variations for finite changes in branch stiffness

Direction (j)	Load P_j	$(TSS)_n$ obtained by changing K_{b3} to 60 kN/mm in the basic system $(\mathrm{TSS})_m$ see Table 7			$\frac{\delta_{km}\,\delta_{kn}}{d_{jm}\,d_{jn}}$	$\dfrac{\frac{1}{(f_{jj})_m} - \frac{1}{(f_{jj})_n}}{\frac{1}{(f_{bk})_m} - \frac{1}{(f_{bk})_n}}$
		$(d_j)_n$	$(\delta_3)_n$	$(1/f_{jj})_n$		
1	100	2.0186	0.2552	49.5402	0.023	0.023
2	200	2.2042	−0.8585	90.7368	0.203	0.203
3	150	1.8794	0.8701	79.8148	0.276	0.276

$(\mathrm{TSS})_m$: $K_{b1} = 10$ kN/mm; $K_{b2} = 20$ kN/mm; $K_{b3} = 30$ kN/mm
$K_{b4} = 40$ kN/mm; $K_{b5} = 50$ kN/mm; $K_{b6} = 60$ kN/mm

8. Design equation based on displacement constraints

Recognizing the need for a good starting point for any optimal iterative design scheme, a systematic method of obtaining such a point is pursued here. It is based on an estimated pair of starting points for the design parameters (which represent a pair of TSS of the model) and utilizes displacement constraints in active directions. It yields a starting point which more or less satisfies all the active displacement constraints.

Finding such a starting point is in itself an optimization problem in the hyperspace defined by branch stiffnesses and limited by the hyperplanes formed by essential constraints. Objective function for which can be defined as the absolute value of the sum of difference in the work done by external forces (a) while going through computed displacements and (b) while going through the corresponding constraint values, computed for all the directions of essential displacement constraints.

Consider a model subjected to external loading, $\{P\}$, which has to satisfy essential displacement constraints, $\{d_{co}\}$ (of order r), in some of the directions. It may be relevant to note that as the maximum displacement occurs in one of the active directions, constraints on displacements in active directions will only be considered as essential. If some of the active direction have no constraints, the vector of displacement constraints is made up by assuming $d_{cj} = d_j$, to facilitate ease in handling and ensuring that $\{d_{co}\}$ is augmented to become $\{d_c\}$ of order 'A' (where 'A' represents the number of active directions). The corresponding optimization problem can be mathematically stated as under:

Objective function:

$$\Delta W = \left| \sum_{j=1}^{A} P_j(d_j - d_{cj}) \right| \tag{40}$$

Constraints:

$$d_j \leqslant d_{cj} \quad j = 1, A \tag{41}$$

where

$$d_j = g(f_b, P)$$

In order to achieve this objective let us define a pair of TSS, viz $(\mathrm{TSS})_m$ and $(\mathrm{TSS})_n$, for the model subjected to the given loading, which may be or may not satisfy the constrains. Let us assume that there exists another system, $(\mathrm{TSS})_r$, a linear objective of the given systems, which represents a near optimum design. In arriving at such a system an assumption is made that each of its branch elements can be obtained on the basis of proportional contributions to the objective function. Mathematically this assumption can be stated as

$$\frac{(\Delta w)_{nrk}}{(\Delta w)_{mrk}} = \frac{(\Delta w)_{nr}}{(\Delta w)_{mr}} \tag{42}$$

where, $(\Delta W)_{nrk}$ and $(\Delta W)_{mrk}$ are defined as the contributions due to branch 'k' to $(\Delta W)_{nr}$ and $(\Delta W)_{mr}$, respectively. Considering pairs of TSS 'n' and 'r' as well as 'm' and 'r', it follows from eqn (38) and eqn (39) that

$$\{1/(f_{bk})_r - 1/(f_{bk})_n\}(\delta_k)_r(\delta_k)_n = (\Delta W)_{nrk} \tag{43}$$

and

$$\{1/(f_{bk})_r - 1/(f_{bk})_m\}(\delta_k)_r(\delta_k)_m = (\Delta W)_{mrk} \tag{44}$$

Dividing eqn (43) by eqn (44) and utilizing eqn (42), we obtain

$$\frac{\{1/(f_{bk})_r - 1/(f_{bk})_n\}}{\{1/(f_{bk})_r - 1/(f_{bk})_m\}}\frac{(\delta_k)_n}{(\delta_k)_m} = \frac{(\Delta W)_{nrk}}{(\Delta W)_{mrk}} = \frac{(\Delta W)_{nr}}{(\Delta W)_{mr}} \tag{45}$$

or

$$\frac{\{1/(f_{bk})_r - 1/(f_{bk})_n\}}{\{1/(f_{bk})_r - 1/(f_{bk})_m\}}\frac{1}{\alpha_k} = \frac{1}{\beta_k} \tag{46}$$

where

$$\alpha = (\delta_k)_m/(\delta_k)_n$$

and

$$\beta = (\Delta W)_{mr}/(\Delta W)_{nr}$$

Solving eqn (46) for $(1/f_{bk})_r$ we obtain the 'Design Equation':

$$(1/f_{bk})_r = (1/f_{bk})_n + [(1/f_{bk})_n - (1/f_{bk})_m][\alpha_k/(\beta - \alpha_k)] \tag{47}$$

With the help of which, values of the branch flexibilities are obtained by choosing $\{d\}_r$ such that displacement values in all the constrained directions are exactly equal to the stipulated constraints. This will liquidate the difference in the work done arising out of partial non-satisfaction of the constraints and thereby yield the optimum value of the objective function.

The above mechanism of approximating the optimum becomes more error prone if some of the active directions are unconstrained. This discrepancy is due to the fact that the contribution to ΔW in eqn (40) for such directions is ignored in the absence of a criterion for computation. This implies that the displacement constraints in the active direction are assumed to be different, while computing ΔW for each of the two systems, which is the root cause for error.

Some essential aspects of design equation

Some of the important facets of Design Equation are as follows:

(1) This method of obtaining an improved starting point commences with an estimated pair of starting points, which while representing the physical model, may or may not belong to the feasible region of the design hyper-space.
(2) The improved design point is a linear combination of the chosen estimated pair of starting points and is obtained with the help of 'Design Equation' [i.e. eqn (47)].
(3) If in a model, flexibility of a branch is desired to be fixed by some other considerations; then choosing identical flexibility coefficient of this branch for the estimated pair of starting points ensures that it would remain the same in the resulting TSS.
(4) If all branch flexibilities of $(TSS)_m$ are proportional to those of $(TSS)_n$, then flexibilities of derived system, $(TSS)_r$, are also proportional to flexibilities of the chosen pair of TSS.

Table 9
Determination of design point $(TSS)_r$

Branch no.	$\{K_b\}_n$	$\{K_b\}_m$	$\{\delta\}_n$	$\{\delta\}_m$	α	$\{K_b\}_r$
1	40.00	100.00	8.3298	3.0137	0.3618	152.2948
2	20.00	65.00	−0.8488	−0.4491	0.5291	86.0315
3	25.00	75.00	2.2812	0.7401	0.3244	129.0372
4	30.00	95.00	−9.7613	−3.3047	0.3386	159.4099
5	35.00	105.00	1.4324	0.2910	0.2032	445.3573
6	50.00	150.00	−7.4801	−2.5646	0.3429	246.6427

$\{P\}^T = \{300 \quad 300 \quad 400\}$ kN
and utilising ΔW from Table 10,
$\beta = (\Delta W)_m/(\Delta W)_n = 0.1685$

Table 10
Evaluation of design equation application

Systems	$(TSS)_n$	$(TSS)_m$	$(TSS)_r$
d_1 value	8.3289	3.0137	1.9365
% error*	(316.5)	(50.7)	(3.17)
d_2 value	7.4801	2.5646	1.5685
% error*	(398.7)	(71.0)	(4.57)
d_3 value	9.7613	3.3047	1.9962
% error*	(388.1)	(65.2)	(0.188)
ΔW	6797.2	1145.3	0.0000

* With respect to their constraint values $\{d_c\}^T = [2.0 \quad 1.5 \quad 1.0]$ mm

Illustration: In order to illustrate the computation of the starting design point, a model conforming to the topological layout given in Fig. 1(A) is utilized. The initial guess values of branch stiffnesses of $(TSS)_n$ and $(TSS)_m$ along with other relevant data and resulting parameters of the system $(TSS)_r$ are given in Table 9.

Effectiveness of application of the design equation is illustrated in Table 10 by providing a comparison of the global displacements and the objective function for all the three systems. Numbers in parenthesis are percentage errors in displacements with respect to the corresponding constraint values. As can be seen that the displacements in $(TSS)_r$ are very close to the required constraint values. In this example, units of stiffness, displacement and ΔW are chosen as kN/mm, mm and Nm, respectively.

9. Conclusions

The simplified theory for analysis and design of structural models (defined in terms of discrete element models with single DOF/node) is presented. A number of theorems and formulae were derived to facilitate:

- Locating maximum displacement directions for eliminating spurious constraints and for aiding conservative preliminary design process by reduction of number of displacement constraints.
- Simplified computation of relative flexibilities (for determination of strong/weak directions in a structure).
- Calculation of external loads corresponding to a prescribed/permissible set of branch forces for design evaluation.
- Determination of upper bounds for global diagonal flexibility coefficients based on branch parameters.
- Evaluation of sensitivity of global flexibility coefficients with respect to branch parameters for carrying out efficient design iterations.

With the help of sensitivity analysis (for finite changes in branch/element parameters), an important equation termed as 'Design Equation' was derived. This provides a good starting design point for problems with active displacement constraints.

The scope of present theorems need extension considering elements other than stiffness and constraints other than on displacements. Yet, the sensitivity theorem has opened up new avenues for evolving a new 'Predicted Correction Algorithm' useful for design and forming an integral part of an overall activity termed as Computer Aided Model Based Design (CAMBD). A pilot version of which has already incorporated these theoretical developments (Prasad et al., 1995) and is in the advanced stage of testing. Where the examples include the design of a shear building consisting of columns (modelled as shear spring) and floors modelled as lumped masses, etc. Further work concerning its extension to 2-D structures involving automobile vehicle structures, modelling is in progress. However, further work is needed for extending the methodology to 3-D structural systems.

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